

# MIDA: A Multi Item-type Double-Auction Mechanism <sup>\*</sup>

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## Abstract

In a seminal paper, McAfee (1992) presented a truthful mechanism for double auction, attaining asymptotically-optimal gain-from-trade without any prior information on the valuations of the traders. McAfee’s mechanism handles single-parametric agents, allowing each seller to sell a single item and each buyer to buy a single item.

In this paper, we present a double-auction mechanism that handles multi-parametric agents and allow multiple items per trader. Each seller is endowed with several units of a pre-specified type and has diminishing marginal returns. Each buyer may buy multiple types and has a gross-substitute valuation function. Both buyers and sellers are quasi-linear in money.

The mechanism is prior-free, ex-post individually-rational, dominant-strategy truthful and strongly budget-balanced. Its gain-from-trade approaches the optimum when the market in all item-types is sufficiently large.

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# 1 Introduction

In the simplest *double auction* a single seller has a single item. The seller values the item for  $s$ , which is private information to the seller. A single buyer values the item for  $b$ , which is private to the buyer. Both buyer and seller have quasi-linear utilities with respect to money. If  $b > s$ , then trade can increase the utility for both traders; there is a potential *gain-from-trade* of  $b - s$ . However, there is no truthful, individually rational, budget-balanced mechanism that will perform the trade if-and-only-if it is beneficial to both traders. The reason is that it is impossible to determine a price truthfully. This is easy to see for a deterministic mechanism. If the mechanism chooses a price  $p < b$ , the seller is incentivized to bid  $(p + b)/2$  to force the price up; similarly, if the mechanism chooses a price  $p > s$ , the buyer is incentivized to force the price down. The impossibility holds even when the valuations are drawn from a known prior distribution and even when the mechanism is allowed to randomize; see the classic papers of Vickrey [44] and Myerson and Satterthwaite [37].

In a seminal work, McAfee [33] showed a way to overcome this impossibility result when there are many sellers, each with a private valuation  $s_i$ , and many buyers, each with private valuation  $b_i$ . In McAfee’s double auction mechanism (slightly simplified for the sake of brevity), each participant is asked to give his valuation. The sellers are sorted in an ascending order according to their valuations  $s_1 \leq s_2 \leq \dots \leq s_n$ , and the buyers are sorted in a descending order  $b_1 \geq b_2 \geq \dots \geq b_n$ . Let  $k$  be the maximal value such that  $s_k \leq b_k$ . The optimal gain-from-trade can be attained by picking any price  $p \in [s_k, b_k]$  and performing  $k$  deals in that price. However, this mechanism is not truthful since the buyers have an incentive to under-bid and the sellers have an incentive to over-bid. Instead, McAfee uses two prices and does  $k - 1$  deals: the sellers with values  $s_1, \dots, s_{k-1}$  sell their item for  $s_k$  and the buyers with values  $b_1, \dots, b_{k-1}$  buy an item for  $b_k$ . This mechanism is truthful since a trading agent cannot affect the price without exiting the trade. Since there are two prices and  $b_k \geq s_k$ , money is left on the table, but at least the mechanism does not need to subsidize the market (such a mechanism is called *weakly budget-balanced*). Also, the mechanism does the  $k - 1$  most beneficial deals out of the  $k$  possible deals, thus achieving a  $1 - 1/k$  approximation to the maximum gain-from-trade.

With all its success, McAfee’s mechanism is limited to the setting where there is only one type of items, and each seller/buyer may sell/buy only a single unit of that type. This paper presents a mechanism that can handle multiple item-types and multiple units per agent. We assume that all agents have *gross-substitute valuations*, i.e., there are no complementarities between item-types (see Section 2 for formal definition). Under this assumption, there always exists a *price-equilibrium* (also called *Walrasian equilibrium*) — a price-vector in which the demand equals the supply. It is known that, in an equilibrium price, it is possible to allocate each buyer/seller a bundle from his demand/supply set such that the gain-from-trade is optimal (see Section 2). However, just like in the single-type-single-unit case, the Walrasian-equilibrium mechanism is not truthful.

Our solution is the MIDA (Multi-Item Double-Auction mechanism), presented in Algorithm 1. Like McAfee’s mechanism, MIDA is *individually rational*, *dominant-strategy truthful* and *prior-free* — it works even for adversarial (worst-case) valuations. Unlike

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**Algorithm 1** *The Multi-Item Double-Auction (MIDA) mechanism*

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**Input:**

- $g$  groups of sellers, each group sells items of the same type. Each seller holds at most  $m$  unit of its type.
- A single group of buyers. Each buyer may want bundles of different item-types, with at most one unit per type.

**Mechanism:**

1. **Valuation Extraction:** Ask each buyer and seller to declare their entire valuation function.
  2. **Random Halving:** Divide the traders to two sub-markets, Right ( $M^R$ ) and Left ( $M^L$ ), by tossing a fair coin for each trader independently of the others.
  3. **Price Calculation:** Based on the declared valuations of the traders in  $M^R$ , calculate a Walrasian-equilibrium price-vector  $p^R$ , and similarly  $p^L$  in  $M^L$  (a Walrasian equilibrium exists under our assumptions — see Section 2).
  - In  $M^L$ , set the price to  $p^R$  and do the following.
    4. **Trader Lottery:** For each item-type  $x$ , order the  $x$ -sellers (the sellers endowed with units of item-type  $x$ ) by a random permutation, so there are  $g$  lines of sellers. Order all buyers in a single line by a random permutation.
    5. **Serial Trade:** Repeatedly: the first buyer in line buys a bundle that maximizes her net utility (based on the declared valuations in Step 1) from the first sellers in the sellers'-lines. Whenever a seller does not want to sell any more units at the specified price, she leaves the market and the next seller in line becomes first.
  - Do steps 4.-5. in market  $M^R$  with price  $p^L$ .
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**Notes**

- In Step 1, we require the traders to declare their entire valuations in advance and execute the optimal trade for them. We do not let them trade for themselves. This is meant to prevent complicated untruthful strategies. For example, suppose a buyer wants to buy two  $x$ -units, but the single  $x$ -seller wants to sell only one unit. If the buyer knows that the seller's gain from selling two units is only slightly less than selling one unit, he can say "either you sell me two units, or I buy nothing at all". We would like to emphasize that such strategies are not allowed — the strategy-space of the traders is only the space of valuations. In other words, we would like to ensure that the game is simultaneous and not sequential.
  - In Step 3, if there are two or more Walrasian equilibria, the mechanism may select one of them arbitrarily. There is positive probability that one market has no buyers or no sellers or no traders at all, but the notion of Walrasian equilibrium is well-defined in these cases too: when there are no buyers, a price-vector of 0 is equilibrium since it makes the supply zero; when there are no sellers, any sufficiently high price-vector is equilibrium since it makes the demand zero; when there are no traders at all, any price-vector is equilibrium since the demand and supply are both equal to zero. In Step 3 we select an arbitrary equilibrium price-vector; this might reduce the gain-from-trade to zero, but the probability of this misfortune is exponentially small and it is covered by our asymptotic analysis.
  - In Step 5., if a buyer is indifferent between two or more bundles, then the auctioneer is allowed to coordinate the tie-breaking in a way that maximizes the gain-from-trade. This is important since an uncoordinated tie-breaking might lead to loss of welfare even with optimal prices [28].
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McAfee’s mechanism, MIDA is *strongly budget-balanced* — all payments are made between buyers and sellers, so that all gain-from-trade is enjoyed by the traders.<sup>1</sup> All these properties are true *universally* (ex-post) — for any outcome of the randomization process.

The rationale of MIDA is simple. The problem underlying the Myerson-Satterthwaite impossibility is that the buyer and seller cannot agree on a price in a truthful manner. We address this problem using *random-sampling*, whereby we halve the market in two, calculate an equilibrium price-vector in the right market and use it in the left market, calculate an equilibrium price-vector in the left market and use it in the right market.

The random-sampling process creates several challenges which we address below.

## 1.1 Material balance

Since the price in each half-market is not exactly an equilibrium price in that half, the number of units demanded by the buyers might differ from the number of units supplied by the sellers. In traditional single-sided auctions, the seller is also the auction designer, and can afford to not sell a small number of items. But in a double auction the auctioneer is only a mediator, so we must have perfect balance. We solve this issue using *random rationing*.<sup>2</sup> In each market, the buyers and the sellers are ordered randomly. Each buyer in turn buys his preferred bundle from the first seller/s and goes home. However, this does not work if both buyers and sellers are multi-type.

**Example 1.** There is a single buyer who wants to buy a car, which can be either Subaru or Fiat. The buyer has unit demand. There is a single seller who can sell both Subaru and Fiat. The seller’s valuation function is additive. The prices are determined exogenously. In the posted prices, the buyer slightly prefers to buy Subaru while the seller slightly prefers to sell Fiat. If the buyer knows the seller’s valuations, he can strategize by saying that he does not want to buy Fiat at all at the market prices. The seller can similarly strategize by saying he does not want to sell Subaru (see Appendix A.1 for a more general negative result).

To overcome this issue, we require that one side of the market be single-type. The single-type side in our mechanism is the sellers’ side, since it is more realistic to assume that each seller is a producer that produces a single type of items, while each buyer is a consumer that may buy different types. However, our mechanism works just as well if the sellers are multi-type and the buyers are single-type.

Even if the sellers are single-type, they might still want to strategize.

**Example 2.** There is a seller holding two units of the same type. His valuation is 1 for a single unit and 10 for two units. The market-price for this item is 6. Hence, the seller’s first-best option is to sell two units, but if this is not possible, the seller prefers to sell nothing than to sell one unit. Therefore, if the seller knows the buyers’ valuations and knows that only one buyer wants to buy in the market prices, he will strategize and say that he does not want to sell anything.

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<sup>1</sup>In contrast, in McAfee’s mechanism the gain-from-trade includes the surplus of the auctioneer. The gain-from-trade of the traders might be arbitrarily close to 0, as we recently showed in [42].

<sup>2</sup>also known as: random serial dictatorship

In the present paper we avoid this issue by assuming that the sellers have weakly *diminishing marginal returns* (DMR). This means that their gain is large when they sell the first unit, and it becomes weakly smaller as they sell more units.<sup>3</sup> Under this assumption, each seller has a weakly-dominant strategy: sell in the posted prices as long as the marginal gain is positive.

## 1.2 Competitive ratio

Our mechanism, like McAfee’s mechanism, aims to maximize the *gain-from-trade* — the net increase in social welfare due to the trade (in a single-type single-unit market, this is exactly the sum of values of the trading buyers minus the sum of values of the trading sellers). Since it is impossible to attain the maximum gain-from-trade in a truthful budget-balanced mechanism [37], the performance of a mechanism is measured by its *competitive ratio* — the ratio between the expected gain-from-trade of the mechanism (expectation taken over the randomization of the mechanism), and the optimal gain-from-trade.

The competitive ratio of McAfee’s mechanism is  $1 - 1/k$ , where  $k$  is the *optimal market size* — the number of deals in the optimal situation. This means that it approaches the maximal gain-from-trade when the market is sufficiently large. The competitive ratio of MIDA similarly depends on the optimal-market-size, but since there are several item-types, there are several market-sizes. For concreteness, we fix an arbitrary Walrasian-equilibrium in the global population and call it “the optimal situation”. We denote by  $k_x$  the total number of units of type  $x$  exchanging hands in the optimal situation, and denote  $k_{\min} := \min_x k_x$ ,  $k_{\max} := \max_x k_x$ . The competitive ratio of MIDA depends on  $k_{\min}$  and  $k_{\max}$ , but it may also depend on several additional factors. We explain the importance of these factors by examples.

The first factors are related to the number of items per trader.

**Example 3.** There is one item-type. All  $k$  efficient deals come from a single seller (a monopolist). In the random-halving process, this seller falls in one of the halves, while in the other half there is no seller. Then the mechanism has no way to determine a reasonable price, and all gain-from-trade might be lost, even if  $k$  is very large.

To avoid this problem, we assume that (a) Each seller can sell at most  $m$  units of its pre-specified type, and (b) there are  $g$  different item-types, and each buyer can buy at most a single unit of each type.<sup>4</sup> The competitive ratio of MIDA depends on  $m$  and on  $g$ .

The next factors are related to the asymmetry between traders and between item-types.

**Example 4.** There are two item-types. One type has high demand and high supply, for example: a pill for treating headaches. The other type has low demand and low supply, for example: a medicine for treating a rare disease. The rare medicine can also cure headache,

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<sup>3</sup>In Appendix B we prove that, for single-type agents, DMR is equivalent to GS. We already assume GS in order to ensure the existence of a price-equilibrium, but it is important to note that we need DMR to ensure truthfulness as well.

<sup>4</sup>We could allow each buyer to buy at most  $m$  units of each type. This would complicate the analysis and not change the results qualitatively.

but the social welfare is much higher if it is used by the rare disease patients. In a free market, this is automatically handled by the price-vector: the price of the rare medicine will be much higher than the price of the headache pills, so that only its patients will want to buy it. However, in our controlled market, due to the sampling error, the price of the rare might be too low. Then, many headache patients might want to buy it. Since the actual allocation is determined randomly, with non-negligible probability most of the rare medicine will be “wasted” on the headache patients and the competitive ratio will be very low (see Appendix A.2 for a concrete numeric example).

There are two alternative ways to avoid this problem. The first way is to bound the asymmetry between the item-types.<sup>5</sup> Specifically, we define the parameter  $c := k_{\max}/k_{\min}$  — the largest ratio between the optimal-market-sizes of different item-types.

An alternative way is to bound the asymmetry between the traders. Specifically, we define the parameter  $h$  as the ratio between the largest to the smallest positive difference between a buyer’s valuation of a single item and a seller’s valuation of the same item. In other words: the agents’ valuations can be normalized such that, for every single item, and for every buyer-seller pair such that the buyer’s valuation is higher than the seller’s valuation, their valuation difference is between 1 and  $h$ . Note that, when all agents have gross-substitute valuations, the marginal utility of an item is always upper-bounded by its utility as a singleton. Hence,  $h$  is an upper bound of the gain-from-trade from a single deal.

In example 4, both  $c$  and  $h$  are large. In this case, a small sampling-error in one item-type can cause a large loss of welfare in another item-type. However, if either  $h$  or  $c$  (or both) are sufficiently small, such problems do not occur.

Our results are summarized in the following theorem.

**Theorem 1.** (Section 4) Consider a double-sided market in which: (a) There are  $g$  item-types; (b) Each seller sells items of a single type, holds at most  $m$  units of that type, and has diminishing marginal returns; (c) Each buyer wants at most one unit of each type, and has gross-substitute valuations.

Then, with probability at least  $1 - o(g/\sqrt{k_{\min} \ln k_{\min}})$ , the expected competitive ratio of MIDA is at least the larger of the following:

$$1 - 2^{3g} \cdot 20mg \cdot c \cdot \sqrt{\frac{\ln^5 k_{\max}}{k_{\max}}} \qquad 1 - 2^{3g} \cdot 20mg \cdot h \cdot \sqrt{\frac{\ln^5 k_{\max}}{k_{\max}}}$$

In particular, when  $m$  and  $g$  are bounded, and either  $c$  or  $h$  (or both) are bounded, and  $k_{\min}, k_{\max}$  are sufficiently large, MIDA approaches the maximum gain-from-trade.

**Remark.** The parameter  $c$  can depend on  $k_{\max}$ . Specifically, if  $c = k_{\max}^r$  where  $r < 0.5$  (so  $k_{\min} \geq k_{\max}^{1-r}$ ) then our competitive ratio is still  $1 - o(1)$ . In Appendix A.2 we show that the analysis is tight in the following sense: if  $k_{\min} < k_{\max}^{0.5}$  (and the parameter  $h$  is also unbounded), then the gain-from-trade of MIDA might not approach 1 even when there are two items  $(x, y)$  and both  $k_x, k_y$  go to  $\infty$ .

<sup>5</sup> This is analogous to the *thickness* condition introduced Kojima and Pathak [31] in the context of two-sided matching.

Although the results with the two parameters look similar, the parameters are unrelated and fundamentally different. Parameter  $c$  involves only the market-sizes of the different item-types in the optimal situation, in accordance with the common practice in the double auction literature since McAfee [33]. It implies that the item-types are sufficiently similar so that, when the market grows, the number of efficient deals in all item-types grow in a similar rate. Parameter  $h$  involves the individual agents. It comes from the random-sampling literature (see Subsection 1.5). There, it is common to assume that the valuation of each buyer with positive valuation is bounded in  $[1, h]$ , so that each buyer contributes at most  $h$  to the total welfare. Our definition is the natural generalization to a two-sided market.

Finally, we note that MIDA does not have to know any of the parameter values in advance. The parameters affect only the analysis.

### 1.3 Negative result

Our analysis requires that the sellers are single-type and allows only the buyers to be multi-type. By symmetry, we could switch the roles of buyers and sellers, allowing the sellers to be multi-type and requiring the buyers to be single-type. However, our mechanism does not work when both sides are multi-type. As a partial justification to this failure, we prove:

**Theorem 2.** (Appendix A.1) When there are  $g$  item-types whose prices are determined exogenously, one additive seller who may sell any type and one unit-demand buyer who may buy any type, the competitive ratio of *any* randomized individually-rational truthful budget-balanced mechanism is at most  $1/g$ .

This theorem can be seen as a strengthening of the above-mentioned Myerson-Satterthwaite impossibility theorem [37]. Myerson-Satterthwaite's impossibility result is valid even when there is a single item-type, but it crucially depends on the fact that the prices are endogenous and it is impossible to agree truthfully on a price. Our Theorem 2 holds even with exogenously-determined prices.

Note that it is trivial to get a competitive ratio of  $1/g$  relative to the exogenously-determined prices: pick an item-type at random and trade this item-type if-and-only-if both the buyer and the seller agree. Our Theorem 2 shows that, when both the buyer and the seller are multi-type, we cannot do better than this trivial mechanism in a truthful way.

### 1.4 Paper layout

The related literature is surveyed below. Section 2 presents the model. Section 3 proves the truthfulness and other strategic properties of the MIDA mechanism. Section 4 analyzes the competitive ratio of MIDA and proves Theorem 1. Section 5 presents some improved competitive-ratio bounds that can be attained in some special cases. Section 6 concludes. The negative results are presented in Appendix A.

Mechanism	Types	Units	Vals	PF	Budget	Competitive ratio
VCG	Many	Many	Any	+	Deficit	1
Trade Reduction (McAfee 1992)	One	One	-	+	Surplus	$1 - 1/k$
Random Sampling (Baliga+ 2003)	One	One	-	-	Balance	$1 - o(1)$
Secondary Market (Xu+ 2010)	One	Many	Add	-	Surplus	$1 - 1/k$
TAHES (Feng+ 2012)	Many	One	-	+	Surplus	Not given
Loertscher+ 2014	One	Many	Any	-	Surplus	$1 - 1/k$
Reallocation (Blumrosen+ 2014)	Many	Many	Sub	-	Surplus	$1 / [8 \cdot \Theta(\log(g))]$
2SPM (Colini+ 2015)	One	One	-	-	Balance	1/4 to 1/16
Comb. 2SPM (Colini+ 2016)	Many	Many	XOS	-	Balance	1/6
MIDA (this paper)	Many	Many	GS	+	Balance	$1 - O(2^{3g} g m \sqrt{\frac{\ln^5 k_{\max}}{k_{\max}}})$

Table 1: Comparison of truthful double-auction mechanisms. Column legend:

- **Mechanism** — see Section 1.5 for details and references.
- **Types** — whether the mechanism can handle different item-types.
- **Units** — whether the mechanism can handle more than one unit per agent.
- **Vals** — what class of agents' valuation-functions can be handled. *Any* is the most general (works for arbitrary valuations). Then *Sub* (sub-additive). Then XOS. Then GS (gross-substitute, a subset of XOS). Then Add (additive) — a subset of GS.
- **PF** — prior-free — whether the mechanism can work for worst-case valuations of the given class.
- **Competitive ratio** — The ratio of the expected gain-from-trade to the optimal gain-from-trade. Higher is better; 1 is best. See Introduction for definition of parameters.

## 1.5 Related Work

Our work combines two lines of research: double auctions and random-sampling mechanisms. We now compare our results to the most relevant results in each line.

### 1.5.1 Double auctions

The double auction research has made many advancements since McAfee [33]. There are variants of McAfee's mechanism for maximizing the auctioneer's surplus [17], handling spatially-distributed markets [2], transaction costs [13], supply chains [1, 3, 4], constraints on the set of traders that can trade simultaneously [15, 21, 46, 49, 50] and online arrival of buyers [45]. All these advancements are still for a single-type single-unit auction. Some remarkable exceptions are detailed below.

1. The Vickrey-Clarke-Grove (VCG) mechanism is a well-known mechanism that can be used in various settings, including double auction with multiple item types and arbitrary valuations. It is dominant-strategy truthful and attains the maximum gain-from-trade. Its main drawback is that it has budget deficit, which means that the auctioneer has to subsidize the trade.

2. Chu and Shen [13] claim that their mechanism for double-auction with transaction costs can handle multiple item-types. However, this depends on substitutability condi-



tions between buyers to buyers, sellers to sellers, and buyers to sellers. It is not clear whether these conditions hold in our setting. Moreover, they do not analyze the performance of their mechanism in the multiple-item case.

3. Feng et al. [22] present TAHES — Truthful double-Auction for HEterogeneous Spectrum. The mechanism is inspired by the problem of re-allocating radio frequency-spectrum from primary owners to secondary users. In their setting, each seller (primary owner) has a single unit of a unique item-type, each buyer (secondary user) has a different valuation for each type, and each buyer wants a single unit. This is similar to a special case of our setting, in which all buyers have unit-demand valuations. Their solution is based on matching the buyers to sellers. Once the buyers and sellers are matched, the buyers are ordered according to their bid-price for the seller matched to them, the sellers are ordered by their ask-price, and McAfee’s trade-reduction mechanism is used. To maintain truthfulness, the matching process is independent of the bids of the agents. This means that it might not attain the maximum gain-from-trade. Indeed, the authors do not provide an analysis of the gain-from-trade of their mechanism (in fact, many double-auction papers related to frequency-spectrum reallocation do not provide a theoretic analysis of their gain-from-trade. Some of them provide simulations based on data specific to the frequency-spectrum domain, and it is not clear how they perform in other domains).

4. Xu et al. [47] present a secondary-market auction. Both buyers and sellers are multi-unit, and the valuations are additive (the value for  $k$  units is  $k$  times the value for a single unit). Moreover, the competitive ratio calculation assumes a prior distribution, so the mechanism is not prior-free. Our mechanism handles any valuations with weakly-diminishing-marginal-returns, including additive valuations as a special case.

5. Loertscher et al. [32] present a multi-unit double-auction. It is weakly-budget-balanced. Apparently, the competitive ratio guarantees depend on a Bayesian prior, so it is not prior-free.

6. Blumrosen and Dobzinski [8] present a *Combinatorial Reallocation* mechanism. It can handle very general settings, including agents that are both buyers and sellers and have arbitrary sub-additive valuations. The competitive ratio is  $1 / [8 \cdot \Theta(\log m)]$  where  $m$  is the maximum number of items per single seller; it does not approach 1 as the market grows. Additionally, their mechanism is not prior-free, since it needs to know the median value of the initial endowment of each seller. There is no budget deficit but there may be a budget surplus.

7. Gonen et al. [25] present a general scheme for converting a truthful mechanism with deficit to a truthful mechanism without deficit. Their scheme can handle combinatorial double auctions, but only when agents are single-valued, e.g, when each agent has a set of desired bundles and values all these bundles the same.

8. Colini et al present several double-auction mechanisms based on posted-prices. All their mechanisms are strongly-budget-balance. Initially, they presented mechanisms for single-type single-unit auctions [15]. Very recently, and contemporaneously to our work, they extended their mechanisms to handle combinatorial auctions [16]. Their work differs from ours in several important aspects. First, they assume a Bayesian prior while we assume a prior-free setting. Second, they approximate the social welfare rather than the gain-from-trade (the social welfare is arguably easier to approximate, since it includes the welfare of non-participating sellers that keep their item). Third, they attain a constant-

factor approximation of the social welfare ( $1/4$ ,  $1/6$  or  $1/16$ ) which does not converge to 1 as the market grows. On the positive side, they handle more general buyer valuations: the buyers may have XOS valuations, which is a superset of submodular valuations, which is a superset of gross-substitute valuations (their sellers are either single-unit or additive).

Table 1 compares our results to some other double-auction mechanisms that we are aware of.

### 1.5.2 Random Sampling

Random sampling is a common technique in mechanism design. It comes in three flavors: *Bayesian*, *prior-independent* and *prior-free*.

1. In the **Bayesian** setting, it is assumed that the agents' valuations are random variables drawn from a known probability distribution. The goal is usually to construct a mechanism that attains optimal expected revenue, expectation taken over the known distribution. The seminal work was done by Myerson [36], who solved the problem for single-parametric agents. His work was extended in many ways. Most relevant to the present paper are the recent works that show that *posted pricing* mechanisms (where prices are determined in advance, independently of the agents' bids) can approximate the revenue of the optimal auction. This was shown for single-parametric agents [9, 39], single unit-demand agent [10] and multiple unit-demand agents [12, 48]. Our work is similar in that, in each half-market, posted-prices are used. However, it is different in that the price calculation does not use a Bayesian prior (since we do not assume any prior); the price is calculated based on the other half-market. Moreover, our welfare guarantees are valid even for worst-case gross-substitute valuations; the expectation is taken over the internal randomization of the mechanism.

2. In the **prior-independent** setting, it is assumed that the agents' valuations are random variables drawn from some unknown probability distribution. Given some samples drawn from this distribution, the challenge is to obtain expected revenue as close as possible to what could be achieved with advance knowledge of the distribution. The main research challenge in this field is to estimate the *sample complexity* — how many samples are needed to attain good performance. The research in this line has advanced from single-unit auctions [14, 20, 29] to general single-parametric agents [34] to multi-parameter unit-demand buyers [18].

We are aware of a single prior-independent random-sampling mechanism for double auctions [7]; it assumes an unknown bounded-support distribution and handles only single-type single-unit traders.

The most recent advancement (which was done contemporaneously to our work) is by Hsu et al. [28]. They study buyers with arbitrary valuations in a single-sided markets with fixed supplies. They prove that, if  $n$  buyers are sampled i.i.d. from some unknown distribution, an optimal price-vector is calculated, and this price-vector is then applied to a fresh sample of  $n$  i.i.d. buyers from the same distribution, then the social welfare is approximately optimal. The competitive ratio implied by their Theorem 6.3 is, with probability  $1 - \delta$ , at least  $1 - O\left(\sqrt{\frac{h^3 n^{0.5} g^4 \ln^2 m \ln^2 \frac{1}{\delta}}{\text{OptimalWelfare}}}\right)$ . Their results are not directly applicable to our problem because their setting is prior-independent while our setting is prior-free

(see below why these settings are different). Moreover, the above bound becomes worse when  $n$  becomes larger, where  $n$  is the total number of agents — including agents that never participate in the trade. In contrast, our results depend only on the numbers of deals in the optimal situation ( $k_x$ ), which is arguably a more meaningful representation of the market-size.

3. In the **prior-free** setting, the agents' valuations are not assumed to come from any probability distribution. This setting is considered more challenging than the prior-independent setting; see e.g. [11].

**Example 5.** There are  $n$  sellers, all of whom have the same high value,  $V$ . For the buyers, consider the following two options:

(a) Buyers' valuations are drawn from a probability distribution such that, with probability  $(n - 1)/n$  their value is 0 and with probability  $1/n$  their value is  $2V$ .

(b) there are  $n - 1$  buyers with value 0 and one buyer with value  $2V$ .

Setting (a) is prior-independent: in two samples of  $n/2$  agents, there is a positive probability that we see the  $2V$  buyer in both samples and set the price to e.g.  $1.5V$ , in which case the gain-from-trade in each market will be  $V$ .

Setting (b) is prior-free: regardless of how the agents are divided to two halves, the  $2V$  buyer will land in one half, so there is no chance of setting the correct price, and the gain-from-trade is always 0.

The random halving technique, used by MIDA, was introduced for prior-free approximation of the maximum revenue in digital goods auctions [23, 24]. It was later extended to handle single-parametric physical-goods environment with various kinds of constraints [19], multiple item-types with infinite supply [5] and finite supply [6]. These extensions are all for a one-sided market, where the supply is fixed in advance. The usual setting involves a seller who wants to sell some supply to small buyers. The seller samples some fraction of the market (usually  $\epsilon$  fraction or half), computes the "optimal" solution on the fraction for some sense of optimality, and applies this solution in the other part of the market. While the seller wants to sell as many units as possible, he is not required to exhaust the supply. In contrast to this literature, in our setting there are two new constraints: (a) *Material balance*: we have to make sure every item sold gets bought; leftover items are not allowed. In Appendix A.1 we use this material-balance requirement to show that, even if prices are determined exogenously, it may be impossible to design an efficient truthful mechanism. (b) Both demand and supply can change, and there is an intricate game between supply and demand. A small sampling-error in the demand-side and supply-side in one item-type can cause a large error in another item-type, and this might lead to a complete market-failure, as illustrated by the example in Appendix A.2.

### 1.5.3 Differences between the double-auction and the random-sampling literature

One difference between these two lines of research is in the definition of a "large" market. In the random-sampling literature, it is common to measure the market size by the number of *potential* traders, usually denoted by  $n$  [5]. Moreover, it is common to assume that the total revenue from a single buyer is bounded by some constant  $h$ , or that buyers are sampled from some unknown bounded-support distribution [7]. In contrast, in

the double-auction literature since McAfee [33], the market size is measured only by the number of *actual* traders in the optimal situation, a number usually denoted by  $k$ . It is possible that  $k \ll n$ , e.g, when there are many “potential” buyers/sellers with low/high valuations, that do not trade even in the optimal situation. Since the present paper is both on random-sampling and on double auctions, we provide results using the two alternative parameters: the market-sizes in different item-types (parameters  $k_{\min}, k_{\max}, c$ ) and the maximum ratio between agents’ values ( $h$ ).

## 2 Preliminaries

### 2.1 Agents and valuations

We consider a market in which some agents, the “sellers”, are endowed with one or more items, and other agents, the “buyers”, are endowed with money. Each agent  $i$  has a valuation-function  $v_i$  on bundles of items. Given a price-vector (a price for each item-type), the agents aim to maximize their *gain*. For a buyer  $i$ , this means buying a bundle  $X_i$  that maximizes the difference  $\text{Gain}_i(X_i) := v_i(X_i) - p(X_i)$ , where  $p(X_i)$  is the sum of the prices of the items in  $X_i$ . For a seller  $j$ , this means selling a bundle  $Y_j$  that maximizes the difference  $\text{Gain}_j(Y_j) := (p(Y_j) + v_j(E_j \setminus Y_j)) - v_j(E_j)$ , where  $E_j$  is the seller’s initial endowment.

We assume that there are  $g$  item-types. Each seller is endowed with at most  $m$  units of a *single* type and has *diminishing marginal returns* (DMR). Formally, if  $v_i(j)$  is the valuation of agent  $i$  for having  $j$  units, then DMR means that  $v_i(j+2) - v_i(j+1) \leq v_i(j+1) - v_i(j)$ , for  $j \in \{0, \dots, m-2\}$ .

Each buyer wants at most a single unit of each item-type. The buyers have *gross-substitute* (GS) valuations [30]. This means that the following is true for every two price-vectors  $p, q$  and every item  $x$  (where  $\Delta_x := q_x - p_x$ ): If  $\forall y : \Delta_y \geq 0$  and  $\Delta_x = 0$  and the buyer wants to buy  $x$  in prices  $p$ , then the buyer wants to buy  $x$  in prices  $q$  (if all items become weakly more expensive while item  $x$  retains its price, then the agent does not cease wanting item  $x$ ).

Given a trading scenario in which each agent  $i$  buys/sells a bundle  $X_i$ , the *gain-from-trade* is the sum of gains of all agents:  $\text{Gain} := \sum_i \text{Gain}_i(X_i)$ . We consider only scenarios which are *materially balanced* — the set of items sold by the sellers equals the set of items bought by the buyers. In such scenarios, the gain-from-trade does not depend on the price, since money is only transferred between buyers and sellers.

### 2.2 Walrasian equilibrium

A Walrasian equilibrium (aka *competitive equilibrium* or *price equilibrium*) consists of a price-vector and an allocation in which the demand and supply are balanced: for each buyer/seller there exists an optimal bundle for buying/selling (a bundle that maximizes the agent’s net utility given the price-vector), such that the union of buyers’ bundles equals the union of the sellers’ bundles. This condition is also called *market-clearing*. Importantly, if all agents have gross-substitute valuations, then: (1) A Walrasian equilibrium

exists, (2) it can be efficiency computed, e.g, using an English auction [27], (3) it attains the maximum gain-from-trade (e.g. Theorem 11.13 in [38]). In our setting, the buyers' valuations are GS and the sellers' valuations are DMR. For single-type agents, DMR and GS are equivalent (Appendix B). Therefore, a Walrasian equilibrium always exists.

### 3 Strategic Analysis

We first prove that MIDA has the same desirable properties of McAfee's mechanism.

**Theorem 3.1.** The MIDA mechanism, presented in Algorithm 1, is prior-free, ex-post individually rational, strongly-budget-balanced, and dominant-strategy truthful.

*Proof.* **Prior-freeness** holds by design.

**Ex-post individual rationality** holds for the buyers since they buy their optimal bundle in the market prices. The sellers leave the market once they have sold their optimal quantity, so they never over-sell. A seller may under-sell if there are not enough buyers in its market who want to buy its item-type in the prices of the other market; however, by our assumption the seller has diminishing-marginal-returns, so selling less than the optimal quantity is weakly better than selling nothing at all.

**Ex-post strong budget-balance** holds since in each market, all trade is between buyers and sellers and the price-vector is the same for buyers and sellers.

**Ex-post dominant-strategy truthfulness** holds in the Serial Trade step since each buyer buys the optimal bundle given the market state. Since the sellers are DMR, it is always optimal for a seller to sell as many units as possible up to the optimal quantity, and then leave the market. This is exactly what the mechanism does for them. In the Price Calculation step, the traders' replies are used only for statistics for the other market, and have no effect on their own market. Moreover, a non-truthful trader might be forced to perform a non-optimal deal in the Serial Trade step. Hence, truth-telling is a weakly-dominant strategy.  $\square$

### 4 Competitive-Ratio Analysis

The remainder of this paper is devoted to the analysis of the gain-from-trade guaranteed by MIDA. We analyze the gain-from-trade in market  $M^L$  when the price is set to  $p^R$ ; the opposite direction (the gain-from-trade in  $M^R$  when the price is set to  $p^L$ ) is completely analogous.

#### 4.1 Notation

The following notation is used throughout the analysis:

$p$  denotes a price-vector.

- $p^O$  is the price-vector in the optimal situation (an arbitrary Walrasian-equilibrium in the global population).

- $p^R$  is the equilibrium price in  $M^R$  (also  $p^L$  is the equilibrium price in  $M^L$ , but our analysis focuses on  $p^R$ ).

Items and bundles:

- Lower-case letters  $x, y, z$  denote item-types.
- Upper-case letters  $X, Y, Z$  denote item-bundles.

Trader sets:

- $B$  denotes a subset of buyers,  $S$  denotes a subset of sellers, and  $D$  denotes a subset of both buyers and sellers.
- Subscripts denote agents' demand/supply in different price-systems. The left subscript relates to the optimal price  $p^O$  and the right subscript to price  $p^R$ . E.g,  $B_{x-}$  is the set of buyers who demand item  $x$  in  $p^O$  but stop wanting  $x$  when the price changes to  $p^R$ ;  $B_{+x}$  is the set of buyers who do not demand  $x$  in  $p^O$  but start wanting  $x$  when the price changes to  $p^R$ .
- Superscript denote sampling, e.g.  $B_{x-}^R$  is the subset of  $B_{x-}$  sampled to  $M^R$  and  $B_{x-}^L$  is the subset sampled to  $M^L$ .

The shorthand "w.p.  $x$ " means "with probability of at least  $x$ ".

## 4.2 Gain-from-trade per item-type

For the analysis, it is convenient to split the total gain-from-trade to a sum of  $g$  terms, one for each item-type. The gain of buyer  $i$  from buying a bundle  $X_i = \{x_1, \dots, x_t\}$  when the price-vector is  $p$  can be presented as:

$$\begin{aligned}
 \text{Gain}_i(X_i) &= v_i(X_i) - p(X_i) \\
 &= v_i(\{x_1\}) - p(x_1) \\
 &\quad + (v_i(\{x_1, x_2\}) - v_i(\{x_1\})) - p(x_2) \\
 &\quad + (v_i(\{x_1, x_2, x_3\}) - v_i(\{x_1, x_2\})) - p(x_3) \\
 &\quad + \dots + (v_i(X_i) - v_i(X_i \setminus \{x_t\})) - p(x_t)
 \end{aligned}$$

Each line of the right-hand side measures the *marginal gain* of the buyer from buying a single item. The total gain of the buyer equals the sum of his marginal gains from all items in  $X_i$ . Note that there are many different ways to split the total gain to marginal gains, depending on the ordering of the items. For our purposes, we fix an arbitrary ordering on the items and denote by  $\text{MarginalGain}_i(x)$  the marginal gain of buyer  $i$  from item  $x$ , given the trading scenario and the fixed ordering. So  $\text{Gain}_i = \sum_{x=1}^g \text{MarginalGain}_i(x)$ .

Since each  $x$ -seller may sell up to  $m$  units of  $x$ , we represent each seller by at most  $m$  "virtual sellers". The gain of a seller is split between its virtual sellers according to their marginal valuation. For example, an  $x$ -seller who values one unit as 7 and for two units as 9 is represented by two single-unit virtual-sellers, a seller  $j1$  with  $v_{j1}(x) = 7$  and a seller  $j2$  with  $v_{j2}(x) = 2$ . If the price-vector is  $p$  then the marginal gain of each virtual-seller  $j$  is  $\text{MarginalGain}_j(x) = p(x) - v_j(x)$ .

The *gain-from-trade in item  $x$*  is defined as the sum of the marginal-gains of all buyers and sellers from this item,  $\text{Gain}(x) := \sum_i \text{MarginalGain}_i(x)$ . By definition, the total gain-from-trade is the sum of gains-from-trade in all item-types:  $\text{Gain} := \sum_{x=1}^g \text{Gain}(x)$ .

### 4.3 Efficient traders

The optimal gain-from-trade in item  $x$  comes from the sets of *efficient traders*, denoted by:

- $S_{x*}$  — the set of sellers who want to sell  $x$  in the optimal equilibrium price,  $p^O$ ;
- $B_{x*}$  — the set of buyers who want to buy  $x$  in price  $p^O$ .

The optimal gain-from-trade from item  $x$  is:

$$\text{OptimalGain}(x) = \sum_{i \in B_{x*} \cup S_{x*}} \text{MarginalGain}_i(x)$$

Since the buyers are multi-type, each buyer may appear in several different sets. For example, a buyer whose demand in  $p^O$  is  $\{x, y\}$  is counted as one buyer in  $B_{x*}$  and one buyer in  $B_{y*}$ . The gain-from-trade is split between these virtual buyers according to their marginal valuation, when the items are ordered by a fixed arbitrary order. For example, a buyer who values  $\{x\}$  as 6,  $\{y\}$  as 8 and  $\{x, y\}$  as 9 can be represented by a virtual-buyer in  $B_{x*}$  with  $\text{MarginalGain}_i(x) = 6 - p(x)$  and a virtual-buyer in  $B_{y*}$  with  $\text{MarginalGain}_i(y) = 3 - p(y)$ . Similarly, each multi-unit seller is represented by several single-unit virtual-sellers with the marginal valuations. For example, an  $x$ -seller whose supply in  $p^O$  is two units, who values one unit as 7 and two units as 9, is represented by two virtual-sellers in  $S_{x*}$ , one with  $\text{MarginalGain}_j(x) = p(x) - 7$  and the other with  $\text{MarginalGain}_j(x) = p(x) - 2$ .

By the definition of  $p^O$  as an equilibrium price, there must be an equal number of efficient virtual-buyers and efficient virtual-sellers. By definition of the market-size  $k_x$ :

$$\forall x \in \{1, \dots, g\} : \quad |B_{x*}| = |S_{x*}| = k_x \quad (\text{Global-Market Clearing})$$

### 4.4 Actual traders

The actual gain-from-trade in item  $x$  comes from the set of virtual-traders who manage to trade in each half-market. These traders may be either efficient traders or inefficient ones. For example, if the price of item  $x$  in  $M^R$  is above its equilibrium price, then all efficient and some inefficient virtual-sellers will want to sell  $x$ . In our analysis, we make the worst-case assumption that the gain-from-trade contributed by inefficient traders is 0. So in our analysis we count only the gain-from-trade that comes from efficient traders.

There are three issues that might harm the gain-from-trade.

- (a) **Sampling error:** the number of efficient virtual-buyers sampled to one half-market may be smaller than the number of efficient virtual-sellers sampled to the same half-market, or vice-versa.
- (b) **Pricing error:** the price in the half-market might be lower than the optimal price so some efficient sellers will not want to sell, or the price might be higher so that some efficient buyers will not want to buy, or the price of *another* item may be lower than the optimal price so that some efficient buyers will switch their demand to that other item.
- (c) **Competition:** the price in the half-market may be lower than the optimal price and bring inefficient buyers that will compete with the efficient ones, or the price may be higher than the optimal price and bring inefficient sellers that will compete with the efficient ones.

In the rest of this paper, we will prove that the number of traders affected by each of these three issues is  $o(k_x)$ . From this we will conclude that the expected competitive ratio is  $1 - o(1)$ .

## 4.5 Sampling error

Standard concentration inequalities imply that, with high probability, the number of traders sampled to a market is approximately half their total number, for example:

$$\text{w.p. } 1 - 2/k_x^2 : \quad \left| |S_{x*}^R| - k_x/2 \right| < e_x \quad (4.1)$$

where  $S_{x*}^R$  denotes the subset of  $S_{x*}$  sampled to market  $M^R$ , and  $e_x$  is an expression bounding the sampling-error. In a single-unit setting,  $e_x = \sqrt{k_x \ln k_x}$  (see e.g. [40]). In a multi-unit setting, the sampling process is done on *real* traders, each of whom represents up to  $m$  *virtual* traders, and these virtual traders are all sampled together. Therefore the sampling-error has an additional factor of  $m$ :

$$e_x = m\sqrt{k_x \ln k_x}$$

Even though our buyers are single-unit, we prefer to simplify the analysis by using the same error factor  $e_x$  both for the buyers and for the sellers. There are bounds analogous to (4.1) for  $|S_{x*}^L|$ ,  $|B_{x*}^R|$  and  $|B_{x*}^L|$ . Combining these inequalities by the union-bound gives that w.p.  $1 - 8/k_x^2$ , in each half-market, at most  $e_x$  efficient virtual-traders are prevented from trading due to the sampling error. The total number of efficient virtual-traders that lose their deal due to sampling is thus at most  $2e_x$ .

## 4.6 Pricing error

The price in each half-market might differ from the optimal price. In the analysis below we focus on  $p^R$  — the price determined by  $M^R$  and applied in  $M^L$ . The other direction is analogous.

The price-distortion does not affect the gain-from-trade directly, since the total gain-from-trade does not depend on the price. For example, if the price of item  $x$  increases, then the gain of  $x$ -buyers decreases but the gain of  $x$ -sellers increases by the same amount, so the total gain does not change. The price-distortion affects the gain-from-trade indirectly, by inducing agents to change their demand and supply. This effect is represented by the following trader sets:

- $S_{x-}$  — virtual-sellers who stop offering  $x$  in  $p^R$  (because its price decreased relative to  $p^O$ ). This creates supply-shortage which might cause efficient  $x$ -buyers to lose deals.
- $B_{x-}$  — virtual-buyers who stop demanding  $x$  in  $p^R$  (because its price increased or the price of another item decreased). This creates demand-shortage which might cause efficient  $x$ -sellers to lose deals.



## 4.7 Competition

The pricing-error induces a third source of loss, represented by the following sets:

- $S_{+x}$  — virtual-sellers who start offering  $x$  in  $p^R$  (because its price increased relative to  $p^O$ ). This creates excess-supply which might cause efficient  $x$ -sellers to lose deals, if the Trader Lottery in Algorithm 1 puts them after the inefficient  $S_{+x}$  sellers.
- $B_{+x}$  — virtual-buyers who start demanding  $x$  in  $p^R$  (because its price decreased or the price of another item increased). This creates excess-demand which might cause efficient  $x$ -buyers to lose deals.

Figure 1 illustrates these trader-sets in a single-type market.

For convenience we combine the trader-sets related to the pricing-error and competition:

- $D_{+x} := B_{+x} \cup S_{x-}$  = the excess-demand caused by the price distortion;
- $D_{x-} := B_{x-} \cup S_{+x}$  = the excess-supply caused by the price distortion.

All in all, the number of virtual-traders that lose their deal in item  $x$  is bounded as:

$$\text{Loss}_x < 2e_x + |D_{+x}| + |D_{x-}| \quad (4.2)$$

All these traders are selected at random from the sets  $B_{x*}$  and  $S_{x*}$  by the Trader Lottery step of MIDA. All traders have gross-substitute valuations, which are, in particular, submodular. Therefore, a lost deal does not decrease the gain-from-trade contributed by other deals.<sup>6</sup> Therefore, the expected loss in gain-from-trade in item  $x$  is at most a fraction  $\text{Loss}_x/k_x$  of the total.

We already know that  $e_x = o(k_x)$ . Our goal in the next sections will be to prove that, with high probability,  $|D_{+x}| = o(k_x)$  and  $|D_{x-}| = o(k_x)$ . This will imply that  $\text{Loss}_x = o(k_x)$  and that the competitive ratio is  $1 - o(1)$ .

## 4.8 Difference between excess supply and excess demand

Above, we used the market-clearing equation on item  $x$  in the global market to derive inequality (4.1). Combining this inequality with an analogous inequality on  $S_{x*}^R$  gives:

$$\text{w.p. } 1 - 4/k_x^2: \quad \left| |B_{x*}^R| - |S_{x*}^R| \right| < 2e_x$$

Now, we use the market-clearing equation on  $x$  in the Right market:

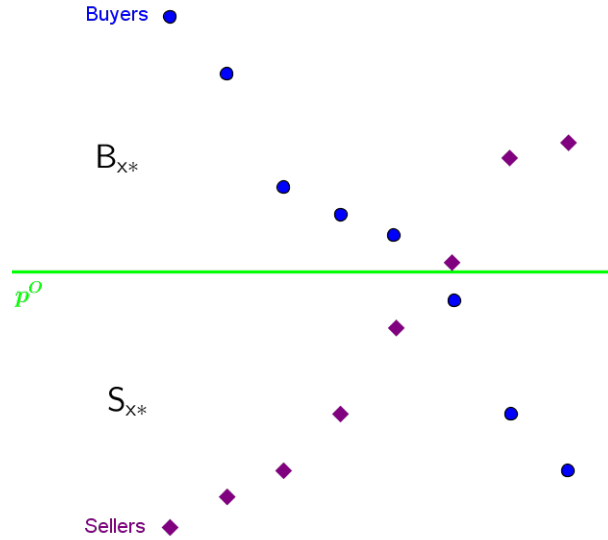
$$|B_{x*}^R| - |B_{x-}^R| + |B_{+x}^R| = |S_{x*}^R| - |S_{x-}^R| + |S_{+x}^R| \quad (\text{Right-Market Clearing})$$

<sup>6</sup>For example, consider an  $x$ -seller who values one unit as 7 and for two units as 9. He is represented in  $S_{x*}$  by two virtual-sellers, one of whom contributes  $p(x) - 7$  and the other  $p(x) - 2$  to the gain-from-trade. If this seller loses one deal, then the gain-from-trade contributed by the other deal is  $p(x) - 2$ , which is the largest of the two contributions.

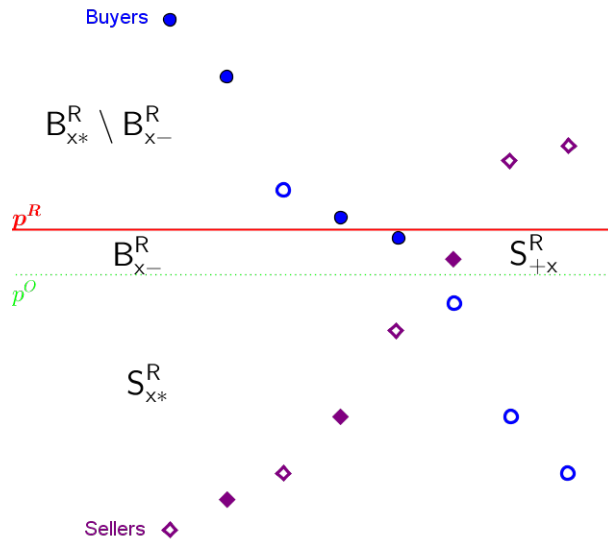
As another example, consider a buyer who values  $\{x\}$  as 6,  $\{y\}$  as 8 and  $\{x,y\}$  as 9. He can be represented by a virtual-buyer in  $B_{x*}$  who contributes  $6 - p(x)$  and a virtual-buyer in  $B_{y*}$  who contributes  $3 - p(y)$  to the gain-from-trade. If this buyer loses the  $y$ -deal, the virtual  $x$ -buyer still contributes  $6 - p(x)$ ; if he loses the  $x$ -deal, the virtual  $y$ -buyer contributes  $8 - p(y)$ , which is even more than  $3 - p(y)$ .

Figure 1: Trader sets in a single-type single-unit market. Balls represent buyers and squares represent sellers; the height of a ball/square corresponds to the agent's monetary valuation of the item. Prices are represented by horizontal lines.

In the global market  $M^O$ , the optimal price is  $p^O$  and  $k_x = 5$ . There are 5 efficient buyers ( $B_{x*}$ ) and 5 efficient sellers ( $S_{x*}$ ).



In the right market  $M^R$ , there are four buyers and three sellers (full balls/squares). The other traders (empty balls/squares) were sampled to the left market. The equilibrium price in the right market is  $p^R$ . In this price, there are 3 buyers who want to buy:  $B_{x*}^R \setminus B_{x-}^R$  ( $\setminus$  is the set difference operator), and 3 sellers who want to sell:  $S_{x*}^R \cup S_{+x}^R$ . Since  $p^R > p^O$ , the trader sets  $B_{+x}$  and  $S_{x-}$  are empty. Note that  $p^R$  is not an equilibrium price in the left market, since there, only 1 buyer wants to buy and 3 sellers want to sell.



The left-hand side is the number of buyers that want  $x$  in  $p^R$  in  $M^R$ , and the right-hand side is the number of sellers that offer  $x$  in  $p^R$  in  $M^R$ . Combining the above two inequalities gives  $\left| (|S_{+x}^R| + |B_{x-}^R|) - (|B_{+x}^R| + |S_{x-}^R|) \right| < 2e_x$ , which can be shortened to:

$$\text{w.p. } 1 - 4/k_x^2: \quad \left| |D_{x-}^R| - |D_{+x}^R| \right| < 2e_x \quad (4.3)$$

We use the union bound to extend this inequality to all item-types simultaneously:

$$\text{w.p. } 1 - 4g/k_{\min}^2: \quad \forall x: \quad \left| |D_{x-}^R| - |D_{+x}^R| \right| < 2e_{\max} \quad (4.4)$$

Inequalities (4.3,4.4) follow directly from the market-clearing conditions, and they are thus true regardless of the agents' valuations. In the following subsections, we deduce from this inequality, a lower bound on the competitive ratio. This is done in three steps:

**Step A:** Move from a bound on a difference of set-sizes,  $\left| |D_{x-}^R| - |D_{+x}^R| \right|$ , to a bound on each set separately,  $|D_{x-}^R|$  and  $|D_{+x}^R|$ .

**Step B:** Move from a bound on these sets in  $M^R$ , to a bound on their parent sets in the global population,  $|D_{x-}|$  and  $|D_{+x}|$ .

**Step C:** Substitute the bounds on  $|D_{x-}|$  and  $|D_{+x}|$  in inequality 4.2 and deduce a bound on the competitive ratio.

## 4.9 Step A: From bound on difference to bound on sets

Inequality 4.4 gives us an upper bound on  $\left| |D_{x-}^R| - |D_{+x}^R| \right|$ . We need upper bounds on  $|D_{x-}^R|$  and  $|D_{+x}^R|$ . We use a recently-proved property of gross-substitute valuations called **Downward Demand Flow (DDF)** [41]. Consider two price-vectors  $p^O$  and  $p^R$ . For every item-type  $x$ , let  $\Delta_x = p_x^R - p_x^O$  be the increase in the price of item  $x$ . Intuitively, DDF means that an agent switches from wanting item  $x$  in price  $p^O$  to wanting item  $y$  in price  $p^R$ , only if  $\Delta_x > \Delta_y$ . This means that the items can be ordered in increasing order of  $\Delta_x$  so that agents switch only downwards (from higher to lower index). Formally:

**Definition 4.1.** A valuation function has the **Downward Demand Flow (DDF)** property if the following are true for every price-vectors  $p^O, p^R$  and for every item-type  $x$ , where  $\Delta_x := p_x^R - p_x^O$ :

- (1) If  $\Delta_x \leq 0$ , then any agent who stopped demanding  $x$ , started demanding  $y$  with  $\Delta_y < \Delta_x$ .
- (2) If  $\Delta_x \geq 0$ , then any agent who started demanding  $x$ , stopped demanding  $y$  with  $\Delta_y > \Delta_x$ .

Segal-Halevi et al. [41] prove that DDF is equivalent to GS. Therefore, all our buyers and sellers have the DDF property, so the following set-inequalities are true:

- (1) For every item  $x$  that became cheaper ( $\Delta_x \leq 0$ ):  $D_{x-} \subseteq \bigcup_{y < x} D_{+y}$ .  
(2) For every item  $x$  that became more expensive ( $\Delta_x \geq 0$ ):  $D_{+x} \subseteq \bigcup_{z > x} D_{z-}$ .

A containment relation on sets trivially holds on their sampled subsets:

**Corollary 4.1** (DDF Corollary). *If all agents have DDF valuations, then:*

- (1) For every item  $x$  that became cheaper:  $D_{x-}^R \subseteq \bigcup_{y < x} D_{+y}^R$ .  
(2) For every item  $x$  that became more expensive:  $D_{+x}^R \subseteq \bigcup_{z > x} D_{z-}^R$ .

See Figure 2 for illustration of the buyer-sets in a unit-demand market.

Suppose the items are numbered  $1, \dots, g$  in increasing order of  $\Delta_x$ . The DDF property implies that, if item 1 became cheaper, then  $D_{1-}^R = \emptyset$ . Applying (4.4) yields that  $|D_{+1}^R| < 2e_{\max}$ . Applying the DDF property again implies that  $|D_{2-}^R| < 2e_{\max}$ . Applying (4.4) again yields that  $|D_{+2}^R| < 4e_{\max}$ . Continuing this way by induction gives us the bounds that we want:

**Lemma 4.1.**

$$w.p. \ 1 - 4g/k_{\min}^2: \quad \forall x: \quad |D_{+x}^R| \leq 2^g \cdot 2e_{\max} \quad \text{and} \quad |D_{x-}^R| \leq 2^g \cdot 2e_{\max} \quad (4.5)$$

*Proof.* We present a proof for items that became cheaper; the proof for items that became more expensive is its mirror image.

We prove by induction on  $x$  — the index of the item-type in the item-ordering — that  $|D_{x-}^R|$  and  $|D_{+x}^R|$  are bounded by  $2^{x-1} \cdot 2e_{\max}$ .

The base is  $x = 1$  — the item with the largest price-decrease. By the DDF Corollary,  $|D_{1-}^R| = 0$ . By Inequality (4.4),  $|D_{+1}^R| < 2e_{\max}$ .

For the induction step, assume the claim is true for all item-types  $y < x$ . By the DDF Corollary:

$$\begin{aligned} D_{x-}^R &\subseteq \bigcup_{y < x} D_{+y}^R \\ \implies |D_{x-}^R| &\leq \sum_{y < x} |D_{+y}^R| < \sum_{y < x} 2^{y-1} \cdot 2e_{\max} = (2^{x-1} - 1) \cdot 2e_{\max} \end{aligned}$$

By Inequality (4.4),  $|D_{+x}^R| < |D_{x-}^R| + 2e_{\max} = 2^{x-1} \cdot 2e_{\max}$ . This concludes the induction proof.

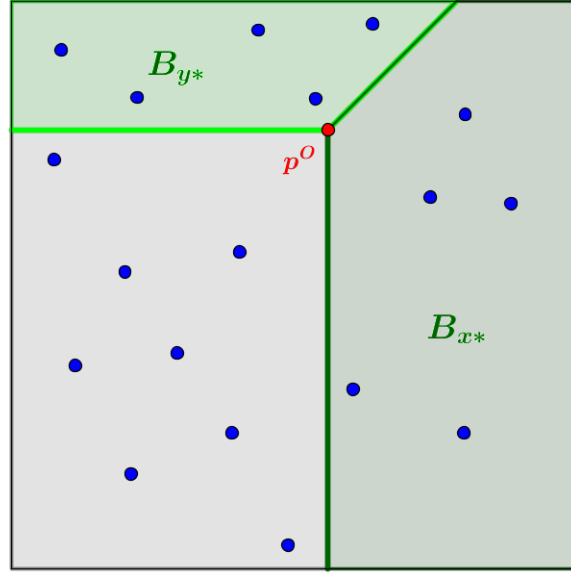
Since  $x - 1 < g$ , the lemma is proved.  $\square$

## 4.10 Step B: from bound on sampled sets to bound on their parent sets

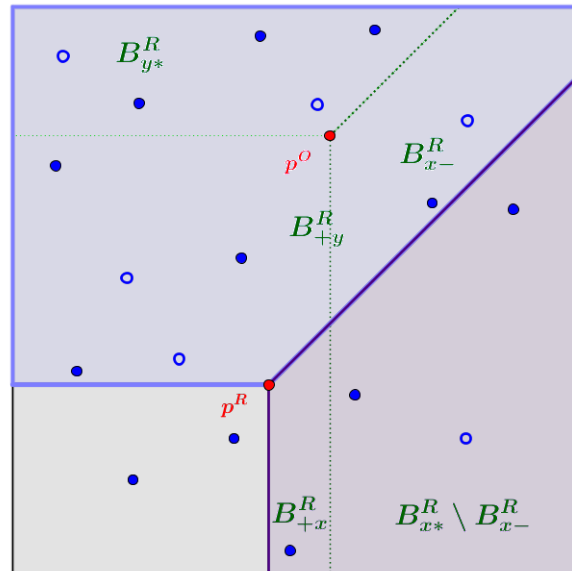
Inequality (4.5) gives us upper bounds on  $|D_{+x}^R|$  and  $|D_{x-}^R|$ . We need upper bounds on  $|D_{+x}|$  and  $|D_{x-}|$ . At first glance, this seems like a simple application of a standard concentration inequality. This is not true, since the standard concentration inequalities apply only to *deterministic sets* — sets that are determined independently of the random-sampling process. The sets  $B_{x*}$  and  $S_{x*}$ , used in Subsection 4.5, are deterministic, because the traders' valuations in the global population are deterministic (we do not assume that they are drawn from any probability distribution). Hence,  $p^O$  is deterministic and so

Figure 2: Buyer sets in a market with two item-types ( $x$  and  $y$ ) and unit-demand buyers. Sellers are not shown. Each buyer is represented by a ball with  $x$ -coordinate  $v_x$  (the buyer's valuation to item  $x$ ) and  $y$ -coordinate  $v_y$  (the buyer's valuation to item  $y$ ). Prices are represented by points  $(p_x, p_y)$ .

In the global market, the optimal price is  $p^O$ . There are  $k_x = 5$  efficient  $x$ -buyers and  $k_y = 5$  efficient  $y$ -buyers (there are also 5 efficient  $x$ -sellers and 5 efficient  $y$ -sellers; they are not shown).



The buyers are randomly divided to two markets. Buyers sampled to the right market are represented below by full balls and the other buyers are empty balls. In the right market, the equilibrium price is  $p^R$ . In this price, there are 7  $y$ -buyers:  $B_{y*}^R \cup B_{+y}^R$  (pentagon to the top-right of  $p^R$ ) and 3  $x$ -buyers:  $B_{x*}^R \setminus B_{x-}^R \cup B_{+x}^R$  (trapezoid to the right-bottom of  $p^R$ ). The 7  $y$ -sellers (in  $S_{y*}^R \setminus S_{y-}^R$ ) and the 3  $x$ -sellers (in  $S_{x*}^R \setminus S_{x-}^R$ ) are not shown. Here, the order of price-changes is  $\Delta_y < \Delta_x < 0$ . Item  $y$  is in the bottom of the ordering, so  $B_{y-} = \emptyset$ . Also, in accordance with the DDF property,  $B_{x-} \subset B_{+y}$ .



are the sets that depend on it. In contrast, the sets  $B_{+x}$  and  $S_{x-}$  (and therefore their union  $D_{+x}$ ) are random, since they depend on  $p^R$ , which in turn depends on the random-sampling process. For such random sets, the standard concentration inequalities cannot be used.<sup>7</sup>

Therefore, we use specialized concentration inequalities that can handle random-sets. These inequalities depend on the *UI dimension* of the random-set. The UI dimension is defined formally in Segal-Halevi and Hassidim [40]. It is related, but not identical, to the *Vapnik-Chervonenkis dimension* [43], which is a fundamental concept in machine learning. For our purposes, it is sufficient to list three rules that can be used to upper-bound the UI dimension. Let  $T$  be a random-set (a random-variable whose possible values are sets). The *support* of  $T$  is defined as the collection of sets that  $T$  can accept with positive probability. The UI dimension of  $T$  is denoted by  $\text{UIDim}(T)$  and can be bounded as follows.

**1. Containment-Order Rule.** If the sets in the support of  $T$  are totally ordered by containment (i.e, for each two possible values of  $T$ , one of them contains the other), then  $\text{UIDim}(T) \leq 1$ . This rule holds for all our seller sets: for every item  $x$ , the set  $S_{x-}$  is the set of virtual-sellers whose valuation for  $x$  is in the range  $[p^R, p^O]$ . Regardless of the value of the random-variable  $p^R$ , the set  $S_{x-}$  contains the  $j$  highest-value sellers from the set of efficient sellers  $S_{x*}$ , for some integer  $j$ . So every possible value of  $S_{x*}$  either contains or is contained in any other possible value. Similar arguments hold for  $S_{+x}$ . Therefore,  $\text{UIDim}(S_{x-}) = \text{UIDim}(S_{+x}) \leq 1$ .

**2. Union Rule.** The UI dimension of a union of random-sets is at most the sum of the UI dimension of the sets:

$$\text{UIDim}(T_1 \cup \dots \cup T_n) \leq \sum_{i=1}^n \text{UIDim}(T_i)$$

**3. Intersection Rule.** The UI dimension of an intersection of random-sets is at most the sum of the UI dimension of the sets, times the logarithm of the *cardinality* of one of the sets. Specifically, if  $T_0$  is a random set whose cardinality is at most  $t$ , then

$$\text{UIDim}(T_0 \cap T_1 \cap \dots \cap T_n) \leq \log(t) \cdot \sum_{i=0}^n \text{UIDim}(T_i)$$

In the special case in which  $T_0$  is a deterministic set, the rule is true with  $\text{UIDim}(T_0) = 0$ .

Lemma 3.5 of Segal-Halevi and Hassidim [40] implies the following concentration-inequality. If  $T$  is a random-set with  $\text{UIDim}(T) \leq d$ , then for every  $t_0 \geq 1$ :

$$\text{w.p. } 1 - 4/t_0 : \quad \text{If } |T| \geq t_0 : \quad \left| |T^R| - |T|/2 \right| < d \cdot m \sqrt{|T| \ln |T|}$$

---

<sup>7</sup>To illustrate, consider a dummy set  $S_{\text{dummy}}$  defined as: "the set of all sellers who were not sampled to  $M^R$ ". Then, by definition  $|S_{\text{dummy}}^R| = 0$  so it is definitely not true that  $|S_{\text{dummy}}^R| \approx |S_{\text{dummy}}|/2$ . We are grateful to Simcha Haber for the explanation.

This inequality is similar to (4.1), except the higher failure probability  $4/t_0$  and the additional factor  $d$  in the upper bound. By simple algebraic manipulations we get the following:

**Lemma 4.2** (Random-Set Concentration Lemma). *Let  $t_0$  be a real number sufficiently large such that*

$$0.1 \cdot t_0 \geq d \cdot m \sqrt{t_0 \ln t_0}.$$

*Then, for every random-set  $T$  with  $\text{UIDim}(T) \leq d$ :*

$$\text{If } |T^R| \leq 0.4 \cdot t_0 \text{ then w.p. } 1 - 4/t_0: |T| < t_0$$

We already have an upper bounds on  $|D_{+x}^R|$  and  $|D_{x-}^R|$  — inequality (4.5). To get the upper bounds on  $|D_{+x}|$  and  $|D_{x-}|$  that we need, all we have to do is to prove that  $D_{+x}$  and  $D_{x-}$  have a bounded UI-dimension. This is what we do in the next subsection.

Our analysis below requires a numeric assumption. Recall that our ultimate goal is to prove Theorem 1, which says that the competitive ratio is at least  $1 - 2^{3g} \cdot 20mg \cdot c \cdot \sqrt{\frac{\ln^5 k_{\max}}{k_{\max}}}$  and at least  $1 - 2^{3g} \cdot 20mg \cdot h \cdot \sqrt{\frac{\ln^5 k_{\max}}{k_{\max}}}$ . These guarantees are meaningful (nonzero) only if the following assumption holds:

$$\frac{\sqrt{k_{\max}}}{(\ln k_{\max})^{5/2}} \geq 20m \cdot 2^{3g} \quad (\text{Assumption 1})$$

so we make this assumption from now on.

## Dimensions of trader-sets

We already explained that the UI dimension of the seller sets  $S_{x-}$  and  $S_{+x}$  is 1. We now prove that UI dimension of the buyer sets  $B_{x-}$  and  $B_{+x}$  is bounded. Define the following auxiliary sets. For every two different bundles  $Y$  and  $Z$  including the empty bundle, let  $\mathbb{B}_{Z \prec Y}$  = the set of buyers who prefer the bundle  $Y$  to the bundle  $Z$  when the price vector is  $p^R$ . All these sets are totally ordered by containment, since they contain all agents for whom  $v_i(Z) - v_i(Y) < p_Z^R - p_Y^R$ . Therefore, for every  $Y, Z$ :  $\text{UIDim}(\mathbb{B}_{Z \prec Y}) = 1$ .

**Lemma 4.3.** *For every item  $x$ ,  $\text{UIDim}(B_{x-}) \leq 2^{2g} \ln k_{\max}$ .*

*Proof.* The set  $B_{x-}$  contains the buyers who want  $x$  in  $p^O$  but do not want  $x$  in  $p^R$ . A buyer does not want  $x$  in  $p^R$ , if-and-only-if for each bundle  $X \ni x$ , there is another bundle  $Y \not\ni x$  such that the buyer prefers  $Y$  over  $X$ . So:

$$B_{x-} = B_{x*} \cap \bigcap_{X \ni x} \left( \bigcup_{Y \not\ni x} \mathbb{B}_{X \prec Y} \right)$$

The set  $B_{x*}$  is deterministic and its size is exactly  $k_x$ . In each union there are  $2^{g-1}$  sets with UI dimension equal to 1, so by the Union Rule, the UI dimension of each union is at most  $2^{g-1}$ . The total number of unions in the intersection is also  $2^{g-1}$ , so by the Intersection Rule, the UI dimension of  $B_{x-}$  is at most  $(2^{g-1} \cdot 2^{g-1}) \ln k_x < 2^{2g} \ln k_{\max}$ .  $\square$

To bound the UI dimension of the sets  $B_{+x}$ , we use a new auxiliary set,  $\mathbb{B}_{any}$  — the set of all buyers who want any non-empty bundle in price  $p^R$ .

**Lemma 4.4.**  $\text{UIDim}(\mathbb{B}_{any}) \leq g$ .

*Proof.*  $\mathbb{B}_{any}$  can be written as the following union:

$$\mathbb{B}_{any} = \bigcup_{Y \neq \emptyset} \mathbb{B}_{\emptyset \prec Y}$$

where the union is over all  $2^g - 1$  non-empty bundles. Moreover, the buyers' valuations are submodular, so for every buyer in  $\mathbb{B}_{any}$  there exists an item  $y$  such that the buyer prefers that item over the empty bundle. Hence  $\mathbb{B}_{any}$  can be written as a much smaller union:

$$\mathbb{B}_{any} = \bigcup_{y=1}^g \mathbb{B}_{\emptyset \prec y}$$

where the union here is over all item-types.  $\mathbb{B}_{any}$  is a union of  $g$  random-sets with UI dimension 1, so by the Union Rule its UI dimension is at most  $g$ .  $\square$

For our purposes, the following loose upper bound on the size of  $\mathbb{B}_{any}$  is sufficient.

**Lemma 4.5.** *With probability 1:*

$$|\mathbb{B}_{any}^R| < 2g \cdot k_{\max}$$

*Proof.* The set  $\mathbb{B}_{any}$  is the union of two subsets: (1) buyers whose demanded bundle in  $p^R$  includes only items that became more expensive, and (2) buyers whose demanded bundle in  $p^R$  includes one or more items that became cheaper.

Every buyer from subset (1) must have wanted some items in  $p^O$ , since these items were more attractive before the price increase. So the number of buyers in subset (1) is at most the total number of deals done in  $p^O$ , which is at most  $g \cdot k_{\max}$ .

By the market-clearance condition, the number of buyers in subset (2) in market  $M^R$  is bounded by the number of virtual-sellers of items that became cheaper in  $M^R$ . The number of these virtual-sellers in the global population is at most  $g \cdot k_{\max}$ , and their number in  $p^R$  cannot be higher since a seller will not start selling an item that became cheaper. Hence, the number of these virtual-sellers in  $M^R$  is at most  $g \cdot k_{\max}$ , and so is the number of buyers in subset (2).

All in all,  $|\mathbb{B}_{any}^R| < 2g \cdot k_{\max}$  as claimed.  $\square$

**Lemma 4.6.** *With probability at least  $1 - 4/\sqrt{(5g \cdot k_{\max}) \ln(5g \cdot k_{\max})}$ :*

$$|\mathbb{B}_{any}| < 5g \cdot k_{\max}$$

*Proof.* Apply the Random-Set Concentration Lemma (4.2) with  $d = g$  and  $t_0 = 5g \cdot k_{\max}$ . By Lemma 4.5,  $|\mathbb{B}_{any}^R| < 0.4t_0$ . By Assumption 1 and some algebraic manipulations, the condition  $0.1t_0 \geq dm\sqrt{t_0 \ln t_0}$  is satisfied.



$$\begin{aligned}
\frac{\sqrt{k_{\max}}}{(\ln k_{\max})^{5/2}} &\geq 20m \cdot 2^{3g} && \implies k_{\max} > 5g \implies 2 \ln k_{\max} > \ln(5g k_{\max}) \\
\frac{\sqrt{k_{\max}}}{(\ln k_{\max})^{5/2}} &\geq 20m \cdot 2^{3g} && \implies \\
\frac{0.5 \cdot \sqrt{k_{\max}}}{\sqrt{2 \ln k_{\max}}} &\geq m \cdot 2^{3g} > m \cdot g \sqrt{5g} && \implies \\
\frac{0.5g \cdot \sqrt{k_{\max}}}{\sqrt{\ln(5g k_{\max})}} &\geq gm \sqrt{5g} && \implies \\
0.5g \cdot k_{\max} &\geq gm \sqrt{(5g k_{\max}) \ln(5g k_{\max})} && \implies \\
0.1t_0 &\geq dm \sqrt{t_0 \ln t_0}
\end{aligned}$$

Hence, w.p.  $1 - 4/\sqrt{(5g \cdot k_{\max}) \ln(5g \cdot k_{\max})}$ , we get the required bound  $|\mathbb{B}_{any}| < t_0$ .  $\square$

**Lemma 4.7.** *W.p.  $1 - 4/\sqrt{(5g \cdot k_{\max}) \ln(5g \cdot k_{\max})}$ , for every  $x$ ,  $\text{UIDim}(B_{+x}) < 2^{2g} \ln k_{\max}$ .*

*Proof.* For every item  $x$ ,  $B_{+x}$  is the set of buyers who want a nonempty bundle in  $p^R$ , want a bundle  $X \ni x$  in  $p^R$ , and do not want  $x$  in  $p^O$ :

$$\begin{aligned}
B_{+x} &= \mathbb{B}_{any} \cap \bigcup_{X \ni x} \left( \bigcap_{Y \not\ni x} \mathbb{B}_{Y \prec X} \right) \cap \overline{\mathbb{B}_{x*}} \\
&= \bigcup_{X \ni x} \left( \mathbb{B}_{any} \cap \bigcap_{Y \not\ni x} \mathbb{B}_{Y \prec X} \right) \cap \overline{\mathbb{B}_{x*}}
\end{aligned}$$

Each intersection inside the big parentheses is an intersection of a bounded random-set with UI dimension at most  $g$ , and  $2^{g-1}$  random-sets with UI dimension 1. By the Intersection Rule, the UI dimension of each intersection is at most  $(g + 2^{g-1} \cdot 1) \cdot \ln(5g \cdot k_{\max})$ . There are  $2^{g-1}$  sets in the union, so by the Union Rule the UI dimension of  $B_{+x}$  is at most  $2^{g-1} \cdot (g + 2^{g-1}) \cdot \ln(5g \cdot k_{\max}) < 2^{2g-1} \cdot \ln(5g \cdot k_{\max})$ .

Assumption 1 implies that  $5g < k_{\max}$ , so the UI dimension of  $B_{+x}$  is less than  $2^{2g} \ln(k_{\max})$ .  $\square$

Now we are finally ready to prove the upper bounds on  $|D_{x-}|, |D_{+x}|$ .

**Lemma 4.8.** *W.p.  $1 - 4/\sqrt{(5g \cdot k_{\max}) \ln(5g \cdot k_{\max})}$ , for every  $x$ , the UI dimension of  $D_{+x}, D_{x-}$  is at most  $2^{2g} \ln k_{\max}$ .*

*Proof.* The sets  $D_{x-}, D_{+x}$  are unions of  $B_{x-}, B_{+x}$  with seller sets whose UI-dimension is 1. Therefore, by the Union Rule, the UI dimension of  $D_{x-}, D_{+x}$  is one plus the UI dimension of  $B_{x-}, B_{+x}$ . By Lemmas 4.3 and 4.7, the latter dimension is at most  $2^{2g} \ln k_{\max} - 1$ .  $\square$

Define:

$$e_{\max} = \max_x e_x = m \sqrt{k_{\max} \ln k_{\max}}$$

**Lemma 4.9.** *W.p.  $1 - 4/\sqrt{(5g \cdot k_{\max}) \ln(5g \cdot k_{\max})} - 4g/k_{\min}^2 - 4/(2^g \cdot 5e_{\max})$ :*

$$|D_{x-}| < 2^g \cdot 5e_{\max} \quad \text{and} \quad |D_{+x}| < 2^g \cdot 5e_{\max} \quad (4.6)$$

*Proof.* Apply the Random-Set Concentration Lemma (4.2) with  $t_0 = 2^g \cdot 5e_{\max} = 2^g \cdot 5m\sqrt{k_{\max} \ln k_{\max}}$  and  $d = 2^{2g} \ln k_{\max}$  (which holds w.p.  $1 - 4/\sqrt{(5g \cdot k_{\max}) \ln(5g \cdot k_{\max})}$ ).

Inequality (4.5) implies that, w.p.  $1 - 4g/k_{\min}^2$ , both  $|D_{x-}^R| \leq 0.4t_0$  and  $|D_{+x}^R| \leq 0.4t_0$ .

By Assumption 1 and some algebraic manipulations, the precondition for using this lemma, namely  $0.1 \cdot t_0 \geq dm\sqrt{t_0 \ln t_0}$ , is satisfied.

$$\begin{aligned}
\frac{\sqrt{k_{\max}}}{(\ln k_{\max})^{5/2}} &\geq 20m \cdot 2^{3g} \implies k_{\max} > 2^g \cdot 5m \\
\frac{\sqrt{k_{\max}}}{(\ln k_{\max})^{5/2}} &\geq 20m \cdot 2^{3g} \text{ and } k_{\max} > 2^g \cdot 5m &\implies \\
\frac{\sqrt{k_{\max}}}{(\ln k_{\max})^{5/2}} &> 20m \cdot 2^{3g} \cdot (\ln(2^g \cdot 5m) / \ln(k_{\max}) + 1) &\implies \\
\frac{\sqrt{k_{\max}}}{(\ln k_{\max})^{3/2}} &> 20m \cdot 2^{3g} \cdot \ln(2^g \cdot 5m \cdot k_{\max}) &\implies \\
\frac{k_{\max}}{(\ln k_{\max}) \sqrt{k_{\max} \ln k_{\max}}} &> 20m \cdot 2^{3g} \cdot \ln(2^g \cdot 5m \cdot k_{\max}) > 20m \cdot 2^{3g} \cdot \ln(2^g \cdot 5m \cdot \sqrt{k_{\max} \ln k_{\max}}) &\implies \\
k_{\max} &> 4 \cdot 5m \cdot 2^{2g} \cdot 2^g \cdot (\ln k_{\max}) \cdot \sqrt{k_{\max} \ln k_{\max}} \cdot \ln(2^g \cdot 5m \sqrt{k_{\max} \ln k_{\max}}) &\implies \\
k_{\max} &> 4 \cdot 2^{2g} \cdot (\ln k_{\max}) \cdot ((2^g \cdot 5m \sqrt{k_{\max} \ln k_{\max}}) \ln(2^g \cdot 5m \sqrt{k_{\max} \ln k_{\max}})) &\implies \\
k_{\max} \ln k_{\max} &> 4 \cdot 2^{2g} \cdot (\ln k_{\max})^2 \cdot (t_0 \ln t_0) &\implies \\
\sqrt{k_{\max} \ln k_{\max}} &> 2 \cdot 2^g \cdot \ln k_{\max} \cdot \sqrt{t_0 \ln t_0} &\implies \\
2^g \cdot 0.5m \cdot \sqrt{k_{\max} \ln k_{\max}} &> m \cdot 2^{2g} \ln k_{\max} \cdot \sqrt{t_0 \ln t_0} &\implies \\
0.1t_0 &> m \cdot d \cdot \sqrt{t_0 \ln t_0}
\end{aligned}$$

Hence, the consequent of the Concentration Lemma is true w.p.  $1 - 4/t_0$ .

By the union bound, we get w.p.  $1 - 4g/k_{\min}^2 - 4/(2^g \cdot 5m\sqrt{k_{\max} \ln k_{\max}}) - 4/(2^g \cdot 5e_{\max})$ :

$$|D_{x-}| < t_0 \quad \text{and} \quad |D_{+x}| < t_0$$

□

## 4.11 Step C: Competitive ratio

Substituting (4.6) in (4.2) implies that, with probability  $1 - o(g/\sqrt{k_{\min} \ln k_{\min}})$ , the expected number of lost deals in each item  $x$  is:  $\text{Loss}_x \leq 2e_x + 2^g \cdot 10 \cdot e_{\max} < 12 \cdot 2^g \cdot e_{\max} = 12 \cdot 2^g \cdot m\sqrt{k_{\max} \ln k_{\max}}$ . The expected number of lost deals overall is at most  $12 \cdot 2^g \cdot mg\sqrt{k_{\max} \ln k_{\max}}$ .

By definition of the parameter  $c$ , the optimal number of deals in each item-type is at least  $k_{\max}/c$ , so the relative loss in each item is at most  $\left(12 \cdot 2^g \cdot mg\sqrt{k_{\max} \ln k_{\max}}\right) / \left(k_{\max}/c\right) = 12 \cdot 2^g \cdot mgc\sqrt{\frac{\ln k_{\max}}{k_{\max}}}$ . Therefore, the expected competitive ratio is at least:

$$1 - 12 \cdot 2^g \cdot mgc \cdot \sqrt{\frac{\ln k_{\max}}{k_{\max}}}$$

Alternatively, by definition of the parameter  $h$ , the contribution of each deal to the gain-from-trade is bounded in the range  $[1, h]$ . The expected loss is thus at most  $12 \cdot 2^g \cdot mgh\sqrt{k_{\max} \ln k_{\max}}$ . On the other hand, the optimal gain-from-trade is at least  $k_{\max}$ .

$1 = k_{\max}$ . Hence, the relative loss is at most  $(12 \cdot 2^g \cdot mgh\sqrt{k_{\max} \ln k_{\max}})/k_{\max}$ , and the competitive ratio is at least:

$$1 - 12 \cdot 2^g \cdot mgh \cdot \sqrt{\frac{\ln k_{\max}}{k_{\max}}}$$

Note that the constants here are much smaller (better) than claimed in Theorem 1. We need the larger constants only for Assumption 1.

## 5 Better Competitive Ratios in Special Cases

The competitive ratio of MIDA improves significantly when the buyers' valuations are restricted. We briefly present two such restrictions. We omit the proofs since they are straightforward adaptations of the ones in Section 4.

**Unit-demand buyers.** If the buyers' valuations are unit-demand (which is a special case of gross-substitute), then with probability  $1 - o(g/\sqrt{k_{\min} \ln k_{\min}})$ , the competitive ratio of MIDA is at least the larger of the following:

$$1 - 640g^2m \cdot c \cdot \sqrt{\frac{\ln^5 k_{\max}}{k_{\max}}} \qquad 1 - 640g^2m \cdot h \cdot \sqrt{\frac{\ln^5 k_{\max}}{k_{\max}}}$$

The main difference from Theorem 1 is that the dependence on  $g$  is polynomial rather than exponential. The reason is that the UI dimension of the buyer sets is smaller. With unit-demand buyers, the sets  $\mathbb{B}_{Z \prec Y}$  are relevant only when the bundles  $Y$  and  $Z$  are singletons. Therefore, each buyer-set is a union/intersection of a linear number of such sets. Using the Union and Intersection rules, it is easy to prove a variant of Lemmas 4.3 and 4.7, in which the UI dimension of the buyer sets is only  $4g \ln k_{\max}$ .

**Single-type buyers.** If there is a single item-type, the buyers (like the sellers) want at most  $m$  units of that type and have diminishing marginal returns, then with probability  $1 - o(1/\sqrt{k \ln k})$ , the competitive ratio of MIDA is at least:

$$1 - 160m\sqrt{\frac{\ln k}{k}}$$

Here, the the UI dimension of the buyer-sets, like that of the seller-sets, is only 1.

## 6 Future Work

The limitations of our mechanism imply two natural directions for future work.

1. Allow both buyers and sellers to be multi-typed. Our example in Appendix A.1 shows why this situation is difficult even when there are exogenously-determined prices.

2. Allow arbitrary valuations rather than just gross-substitute. This poses two challenges: (a) a Walrasian equilibrium might not exist, (b) the Downward-Demand-Flow property might not hold. We do not know how to overcome challenge (a). However, if a Walrasian equilibrium does exist and there are at most three item types, then we can prove that the MIDA mechanism attains asymptotically-optimal competitive ratio for arbitrary submodular valuations, even without the DDF property. We plan to check whether this is also true for four or more item types.

# APPENDIX

## A Negative Results

This appendix contains some examples that illustrate the limitations of ours and other mechanisms.

### A.1 It is not enough to know the prices

The Myerson-Satterthwaite impossibility result involves one buyer, one seller and one item. They prove that it is impossible to extract the maximum gain-from-trade in a truthful way, and the main reason is that it is impossible to truthfully agree on a price.

In our setting, the price is determined exogenously. With a single item-type, it is trivial to extract the maximum gain-from-trade truthfully, since the maximum gain-from-trade is positive if-and-only-if both the buyer and the seller agree to trade. We show that, when there are multiple item-types, it is again impossible to extract the maximum gain-from-trade.

In our setting, there is one buyer and one seller. There are  $g$  items of different types. There is a pre-determined price-vector  $p$ ; the price of each item  $x \in \{1, \dots, g\}$  is  $p_x$ . The seller has an additive valuation function and the buyer has a unit-demand valuation function.<sup>8</sup> The seller's valuation to each item  $x$  is denoted by  $s_x$  and the buyer's valuation is denoted by  $b_x$ .

There is potential trade in at most a single item. We consider only ex-post strongly-budget-balanced trade (all payments are from the buyer to the seller) and ex-post individually-rational trade (the net utility of both agents should be weakly positive). If the trade is in item  $x$ , then the seller's net utility is  $p_x - s_x$ , the buyer's net utility is  $b_x - p_x$  and the gain-from-trade is  $b_x - s_x$ . We define an additional "null item" 0, for which  $p_x = b_x = s_x = 0$ . Trading the "null item" means not trading at all, in which case the utility of both sides and the gain-from-trade are 0.

The optimal gain-from-trade is defined as:

$$OPT := \max_{x: b_x \geq p_x \text{ and } p_x \geq s_x} b_x - s_x$$

This is the maximum gain-from-trade attainable in the current price-system; the maximum gain-from-trade in an item in which both sides are willing to trade (the maximization includes the null item).

Our goal is to approximate the optimal gain-from-trade using a dominant-strategy truthful mechanism. It is easy to get a competitive ratio of  $1/g$  by the following mechanism: select an item-type at random; ask the agents if they are willing to trade in that item; perform the trade if both agents agree. We have a chance of  $1/g$  to select the item responsible for the optimal gain-from-trade, so the expected gain-from-trade is at least

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<sup>8</sup>If both sides are additive, then it is possible to treat the market as  $g$  markets, one for each item-type, and perform the optimal trade in each item-type independently of the others. So to get an impossibility result we must assume that at least one side is not additive.

$OPT/g$ . This mechanism is ex-post truthful. Below, we show that this trivial mechanism is worst-case optimal, even if we are only interested in ex-ante truthfulness.

We are going to prove the following (Theorem 2):

Any ex-ante truthful, ex-post individually-rational, ex-post strongly-budget-balanced mechanism with  $g$  item-types attains an expected competitive ratio of at most  $1/g$ .

A *mechanism* in our model a function  $r$  that receives as input the reported valuations of the agents,  $b' = (b'_1, \dots, b'_g), s' = (s'_1, \dots, s'_g)$ . These reports may be different than the true valuations  $b = (b_1, \dots, b_g), s = (s_1, \dots, s_g)$ . The output of the mechanism,  $r(b', s')$ , is a vector of  $g + 1$  probabilities — a probability for each item-type (including null). The probability of trading item  $x$  is denoted by  $r(b', s'; x)$ , and  $\sum_{x \in \{0, \dots, g\}} r(b', s'; x) = 1$ .

The expected gain of the buyer is:

$$B(b', s') = \sum_{x \in \{0, \dots, g\}} r(b', s'; x) \cdot (b_x - p_x)$$

and the expected gain of the seller is:

$$S(b', s') = \sum_{x \in \{0, \dots, g\}} r(b', s'; x) \cdot (p_x - s_x)$$

and the expected gain-from-trade is:

$$G(b', s') = \sum_{x \in \{0, \dots, g\}} r(b', s'; x) \cdot (b_x - s_x)$$

For our impossibility result, we construct valuations with the following parameters:

- $\epsilon$  — a positive constant such that  $\epsilon \ll 1$ .
- $d$  — an integer between 1 and  $g/2$  (for simplicity we assume that the number of items,  $g$ , is even).
- $p$  — a price vector, with  $p_x \geq 1$  for all  $x \in \{1, \dots, g\}$ .

The buyer is unit-demand and has the following valuations for single items:

- For all  $x \leq d$ :  $b_x - p_x = \epsilon$ .
- For all  $x > d$ :  $b_x - p_x \leq \epsilon \cdot (b_{x-1} - p_{x-1})$ .

In other words, the buyer gains  $\epsilon$  from buying each of the first  $d$  items, and has an at-least-exponentially decreasing gain from buying the other  $g - d$  items (so  $b_{d+1} - p_{d+1} \leq \epsilon^2$ ,  $b_{d+2} - p_{d+2} \leq \epsilon^3$ , etc).

The seller is additive and has the following valuations for single items:

- For all  $x > g - d$ :  $p_x - s_x = \epsilon$ .
- For all  $x \leq g - d$ :  $p_x - s_x \leq \epsilon \cdot (p_{x+1} - s_{x+1})$ .

So the seller gains  $\epsilon$  from selling each of the last  $d$  items, and has an at-least-exponentially decreasing gain from selling the other  $g - d$  items.

We are interested in the probability that the buyer manages to buy one of his desired items,  $1, \dots, d$ :

$$r_d^{buyer}(b, s) := \sum_{x=1}^d r(b, s; x)$$

The probability that the seller manages to sell one of his desired items,  $(g - d + 1), \dots, g$ , is:

$$r_d^{\text{seller}}(b, s) := \sum_{x=g-d+1}^g r(b, s; x)$$

**Lemma A.1.** *Let  $r$  be a DSIC and IR mechanism with competitive ratio  $\alpha$ .*

*Suppose the buyer and the seller have valuations as above with  $d = 1$ .*

*Then, the buyer and seller have a probability of at least  $\alpha$  to trade in one of their  $d$  favorite items:*

$$\begin{aligned} r_1^{\text{buyer}}(b, s) &\geq \alpha \\ r_1^{\text{seller}}(b, s) &\geq \alpha \end{aligned}$$

*Proof.* Consider the alternative valuation  $b'$  which is identical to  $b$  except that the buyer's valuation is such that  $b'_1 - p_1 = 1$  (instead of  $\epsilon$ ). In this valuation, almost all gain-from-trade comes from buyer  $i$  buying item 1. The optimal gain-from-trade is  $\approx 1$  and the mechanism's gain-from-trade is  $\approx r_1^{\text{buyer}}(b, s) \cdot 1$ , since the gain from all other deals is negligible. Hence, the mechanism must ensure that  $r_1^{\text{buyer}}(b, s) \geq \alpha$ . If the true valuation of the buyer is  $b_i$  but he deviates and reports  $b'_i$ , his expected gain is at least  $\alpha \cdot \epsilon$ . Hence, a truthful mechanism must guarantee that the buyer receives at least the same gain by reporting truthfully  $b$ . When  $d = 1$ , almost all the gain of  $b$  comes from buying item 1, so the mechanism must ensure that  $r_1^{\text{buyer}}(b, s) \geq \alpha$ .

By analogous considerations for the seller,  $r_1^{\text{seller}}(b, s) \geq \alpha$  too.  $\square$

**Lemma A.2.** *Let  $r$  be a DSIC and IR mechanism with competitive ratio  $\alpha$ . Then, the buyer and seller have a probability of at least  $d \cdot \alpha$  to trade in one of their  $d$  favorite items:*

$$\begin{aligned} r_d^{\text{buyer}}(b, s) &\geq d \cdot \alpha \\ r_d^{\text{seller}}(b, s) &\geq d \cdot \alpha \end{aligned}$$

*Proof.* By induction on  $d$ . The basis is Lemma A.1. We assume that the claim is true for  $d - 1$  and prove that it is also true for  $d$ . Consider the alternative valuation  $b'$  which is identical to  $b$  except for the buyer's valuation, which is:

- For all  $x \leq d - 1$ :  $b'_x - p_x = \epsilon$ .
- $b'_d - p_d = 0$ .
- For all  $x > d$ :  $b'_x - p_x = -1$ .

In other words, the buyer gains  $\epsilon$  from buying each of the first  $d - 1$  items, gains nothing from buying item  $d$  and loses from buying each of the other  $g - d$  items. By the induction assumption, the buyer has a probability of at least  $(d - 1) \cdot \alpha$  to buy one of his  $d - 1$  favorite items:

$$r_{d-1}^{\text{buyer}}(b', s) = \sum_{x=1}^{d-1} r(b', s; x) \geq (d - 1) \cdot \alpha$$

By ex-post-IR, the mechanism must have  $r(b', s; x) = 0$  for all  $x > d$ . Hence, almost all the seller's gain now comes from selling item  $d$  (the gain from trading items  $1, \dots, d - 1$

is smaller by a factor of at least  $\epsilon$ ). By an argument similar to Lemma A.1, the mechanism must have  $r(b', s; d) \geq \alpha$  (otherwise the seller will deviate to  $s'$  in which e.g.  $s'_d = 1$ ). To summarize: if the buyer's true valuation is  $b'$ , then  $\sum_{x=1}^d r(b', s) \geq (d-1) \cdot \alpha + \alpha = d \cdot \alpha$ .

If the buyer's true valuation is  $b$  but he deviates to reporting  $b'$ , his gain is thus at least  $d \cdot \alpha \cdot \epsilon$ . Hence, a truthful mechanism must guarantee that the buyer receives at least the same gain when telling the truth. Since almost all the buyer's gain comes from buying items  $1, \dots, d$ , the mechanism must make sure that  $\sum_{x=1}^d r(b', s) \geq d \cdot \alpha$  too.

Since there is only one buyer and one seller, the seller also sells a single item, so by analogous considerations,  $\sum_{x=g-d+1}^g r(b', s) \geq d \cdot \alpha$ .  $\square$

of Theorem 4. Apply Lemma A.2 with  $d = g/2$ . The probability that the traded item is one of  $1, \dots, g/2$  is  $(g/2) \cdot \alpha$  and the probability that it one of the other items is also  $(g/2) \cdot \alpha$ . Since the sum of probabilities is at most 1,  $g \cdot \alpha \leq 1$  so  $\alpha \leq 1/g$ .  $\square$

## A.2 Loss of welfare due to demand-supply interaction

Our analysis requires one of two assumptions (see the Introduction): either  $k_{\min} \geq k_{\max}^{1-r}$  for some constant  $r < 0.5$ , or the positive differences between buyers' valuations and sellers' valuations are bounded in  $[1, h]$  where  $h$  is a constant independent of the market size.

The following example shows that these requirements are real and not only an artifact of our analysis. The example shows that, when both assumptions do not hold, there is a constant probability of losing most of the gain-from-trade, so the approximation ratio does not converge to 1.

There are five sets of traders in the market. Their sizes and valuations are presented in the following table, where  $k$  and  $K$  are parameters and  $K \geq k$ .

Name	# $M^O$	# $M^R$	# $M^L$	Value x	Value y	In $p^O$ : $6 \leq p_x^O \leq p_y^O \leq 9$	In $p^R$ : $1 = p_y^R \leq p_x^R \leq 6$
$B_{yy}$	$2k^2$	$k^2$	$k^2$	0	9	Buy y	Buy y
$B_{xy}$	$2k - 2$	$k - 1$	$k - 1$	9	9	Buy x	Buy y
$B_{xx}$	2	0	2	$k^{100}$	0	Buy x	Buy x
$S_{yy}$	$2k^2$	$k^2 + K$	$k^2 - K$	—	1	Sell y	Indifferent
$S_{x\emptyset}$	$2k$	$k$	$k$	6	—	Sell x	Pass

The columns #  $M^O$ , #  $M^R$  and #  $M^L$  show the number of traders of each set in the global population and in the two half-markets, respectively. Note that the sampling deviation in each set is well within the probable limits since:

- Most sets are sampled exactly half-by-half, which is the ideal situation for any sampling mechanism.
- The two members of  $S_{xx}$  fall in  $M^R$ ; this can happen with probability  $1/4$ .
- In  $S_{yy}$ , the sampling deviation is at least  $\sqrt{S_{yy}}$ ; this can happen with constant probability.

In all cases, the probability of deviation is independent of  $k$ , while the number of efficient deals in *both* item-types is increasing with  $k$ , which can be arbitrarily large.



In the global market, the optimal prices are  $6 \leq p_x^O \leq p_y^O \leq 9$ . These prices ensure that all sellers are willing to sell, all buyers are willing to buy, and the supply of  $x$  and  $y$  is divided between the buyers according to their demand (so the  $B_{xy}$  buyers buy  $x$  from the  $S_{x\emptyset}$  sellers).

In  $M^R$ , there is a deviation in the number of  $S_{yy}$  sellers. This deviation is small relative to their total number, but large relative to the number of  $x$  buyers. To balance the excess supply, the price of  $y$  goes down until the  $B_{xy}$  buyers start buying  $y$  and the  $S_{yy}$  sellers are indifferent between selling and not selling. Moreover, the two  $B_{xx}$  sellers happen to fall in the other market. Hence, there is no demand at all for  $x$ , so in equilibrium the price of  $x$  goes down until the  $S_{x\emptyset}$  sellers are out of the market. If we assume that the mechanism takes the minimal Walrasian price, this price will be 1, so that the  $B_{xy}$  buyers agree to buy  $y$ .

In  $M^L$ , the price of  $x$  is too low so the  $S_{x\emptyset}$  sellers do not sell. The supply of  $x$  is 0 and the two important  $B_{xx}$  buyers cannot buy. Almost all welfare is lost. This is summarized in the table below.

Item	Deals in $M^O$	Deals in $M^R$	Demand in $M^L$	Supply in $M^L$
$y$	$B_{yy} = 2k^2 = S_{yy}$	$B_{yy}^A + B_{xy}^A = k^2 + k - 1 < S_{yy}^A$	$B_{yy}^B + B_{xy}^B = k^2 + k - 1$	$S_{yy}^B = k^2 - K$
$x$	$B_{xx} + B_{xy} = 2k = S_{x\emptyset}$	$B_{xx}^A = 0$	$B_{xx}^B = 2$	0

Note that this disaster cannot happen with one item-type, even when almost all welfare comes from a single buyer. The reason is that, as long as  $k$  (the number of efficient deals) is sufficiently large, there will be sufficiently many sellers willing to sell the item in a price that the important buyer is willing to pay.

## B Equivalence of GS and DMR with a single item-type

In the paper, we assumed that the buyers have gross-substitute (GS) valuations, and that the sellers are single-type with decreasing-marginal-returns (DMR) valuations. In this section we prove that, when there is a single item-type, DMR and GS are equivalent.

**Lemma B.1.** *Suppose an agent can have only items of a single type, so that the valuation of each bundle depends only on the number of items in the bundle.*

*Then, a valuation  $v$  is gross-substitute if-and-only-if it has decreasing-marginal-returns.*

*Proof.* If  $v$  is GS, then it is submodular. Gul and Stacchetti [26] prove that submodularity is equivalent to DMR.

Conversely, suppose  $v$  has DMR. We prove that it has *strong-no-complementaries* (SNC); recently, Murota [35] proved that SNC is equivalent to GS.

The SNC condition says that, for all bundles  $X, Y$  and for every subset  $X' \subseteq X$ , there is a subset  $Y' \subseteq Y$  such that:

$$v(X) + v(Y) \leq v(X \setminus X' \cup Y') + v(Y \setminus Y' \cup X')$$

When there is a single item-type, the value of a bundle depends only on the number of items in it, so SNC is equivalent to the following condition: for all integers  $k_x, k_y$  and for every  $k'_x \leq k_x$ , there is an integer  $k'_y \leq k_y$  such that:

$$v(k_x) + v(k_y) \leq v(k_x - k'_x + k'_y) + v(k_y - k'_y + k'_x)$$

Indeed, if  $k'_x \leq k_y$  then we can take  $k'_y = k'_x$  which makes the two sides identical; if  $k'_x > k_y$  we can take  $k'_y = k_y$  which makes the inequality:

$$\begin{aligned} u(k_x) + u(k_y) &\leq u(k_y + k_x - k'_x) + u(k'_x) \\ \iff u(k'_x + [k_x - k'_x]) - u(k'_x) &\leq u(k_y + [k_x - k'_x]) - u(k_y) \end{aligned}$$

The last inequality follows immediately from DMR because  $k'_x > k_y$ . □

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